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APPLICATION OF A VARIATIONAL FORMULATION TO NONEQUILIBRIUM FLUID FLOW

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Abstract—The recent papers of Glansdorff, Prigogine and Hays have shown that a variational principle may be applied to problems in fluid flow. As examples of the use of a variational formulation, the problems of slow viscous incompressible flow between parallel plates and in a circular tube are solved for the case where the phenomenological coefficients of thermal conductivity and viscosity are functions of temperature. The method of Rayleigh–Ritz is used with the variational form to obtain solutions which are compared with solutions obtained by direct analytical techniques. Close agreement between the two methods of analysis is obtained for both Couette and Poiseuille flow, thus establishing a measure of confidence in the variational solution to problems for which direct solutions cannot be obtained.

NOMENCLATURE

B, constant in assumed velocity profile;
C, integration constants;
K, dimensionless temperature ratio;
J, functional;
k, thermal conductivity;
n, unit normal to surface;
P, pressure;
r, radius (cylindrical coordinates);
T, temperature (absolute);
u, v, velocity;
w, heat flux;
X, body force;
x, y, z, Cartesian coordinates;
V, volume.

Greek symbols
α, β, γ, constants;
Γ, flow parameter;
Δ, temperature difference;
δ, variation notation;
ϵ, dummy variable;
η, dimensionless temperature;
θ, angle (cylindrical coordinates);
λ, dimensionless temperature ratio, Lagrangian multiplier;
μ, viscosity (dynamic);
ξ, dummy variable;
Π, negative definite forms;
ρ, ϕ, dimensionless space coordinates;
χ, flow parameter;
Ω, surface;
ω, potential for body force.

Subscripts
i, j, tensorial indices;
1, wall property;
o, reference state;
m, maximum.

Superscripts
$\bar{\ }$, bar (dimensionless notations for temperature and velocity).

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1. INTRODUCTION

THE PROBLEMS of nonisothermal fluid flow are highly nonlinear even when the principal nonlinear terms such as the inertial and convection terms are neglected. An important step in the development of a variational form which can be used to provide good approximate solutions to nonisothermal fluid flow problems was the development of a variational formulation by Glandsdorff, Prigogine and Hays [1] in which the theorem of minimum entropy production [2] was extended to include the effects of mechanical irreversibility due to viscous forces.

The work of Glansdorff *et al.* was limited to the case of a slowly moving incompressible fluid with a nonuniform temperature. The variational formulation was derived from one of two negative definite forms, the first of which

$$\pi_1 = T^{-1}\frac{\partial T}{\partial t}\frac{\partial w_i}{\partial x_i} + T^{-1}\frac{\partial T}{\partial t}P_{ij}\frac{\partial v_i}{\partial x_j} - 2\rho X_i\frac{\partial v_i}{\partial t} + \frac{2\partial P_{ij}}{\partial x_i}\frac{\partial v_i}{\partial t} \leqslant 0 \qquad (1.1)$$

was extended into the variational form

$$\delta\int_V\left\{\frac{1}{2}\left[\frac{k_o T_o}{T^2}\left(\frac{\partial T}{\partial x_i}\right)^2 + \frac{\mu_o T_o}{T}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2\right] + \lambda\frac{\partial v_i}{\partial x_i}\right\}\mathrm{d}V + \int_\Omega\left[\frac{-k_o T_o}{T^2}\left(\frac{\partial T}{\partial x_i}\right)\delta T + 2P_{ij}\delta v_j - 2\rho\omega\delta v_i\right]n_i\,\mathrm{d}\Omega = 0 \qquad (1.2)$$

in which the viscosity and thermal conductivity were of the form

$$\mu = \frac{\mu_o T_o}{T} \qquad k = \frac{k_o T_o}{T} \qquad (1.3)$$

where the subscript "o" emphasizes evaluation of the phenomenological coefficients at a reference temperature T_o. The form given by equation (1.2) has been used by Hays [3] to solve the problems of Couette and Poiseuille flow. The second negative definite form given by

$$\pi_2 = \frac{2}{T}\frac{\partial T}{\partial t}\frac{\partial w_i}{\partial x_i} + \frac{2}{T}\frac{\partial T}{\partial t}P_{ij}\frac{\partial v_i}{\partial x_j} - 2\rho X_i\frac{\partial v_i}{\partial t} + 2\frac{\partial P_{ij}}{\partial x_j}\frac{\partial v_i}{\partial t} \leqslant 0 \qquad (1.4)$$

can be extended into the formulation

$$\delta\int_V\left\{\left[\frac{w_i}{T}\frac{\partial T}{\partial x_i} + (\delta_{ij}P - P_{ij})\frac{\partial v_i}{\partial x_j}\right] + \lambda\frac{\partial v_i}{\partial x_j}\right\}\mathrm{d}V + \int_\Omega\left[\frac{2w_i}{T}\delta T + 2(P_{ij}\delta v_j - \rho\omega\delta v_i)\right]n_i\,\mathrm{d}\Omega = 0 \qquad (1.5)$$

where the phenomenological coefficients

$$k = \frac{k_o T_o}{T} \qquad \mu = \frac{\mu_o T_o^2}{T^2} \qquad (1.6)$$

are employed. Since the theoretical form (1.5) has not been applied to any specific problem, it seemed appropriate to investigate the problems of Couette and Poiseuille flow, as was done previously by Hays [3]; these problems are amenable to analysis by classical analytical techniques and may be compared with the variational solutions.

2. POISEUILLE FLOW

Picard's solution

To determine the accuracy of the variational solution for Poiseuille flow through a circular tube, an analytical solution is required which has as a basis the energy and momentum equations. The geometry of the problem lends itself to cylindrical coordinates as shown in Fig. 1.

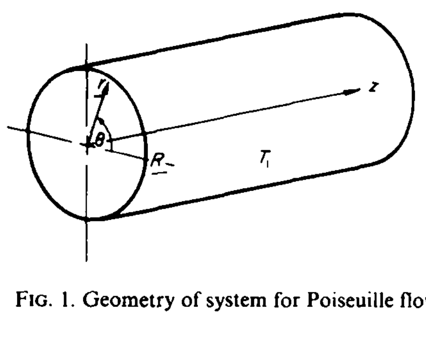

FIG. 1. Geometry of system for Poiseuille flow.

The temperature of the tube wall is maintained at a constant temperature T_1, thus any temperature gradient observed is due to the effects of viscous dissipation within the system. The radial pressure gradient is zero, and dP/dz is constant. From symmetry, the temperature and velocity distributions are functions of r only. The unidimensional momentum and energy equations become

momentum

$$\frac{dP}{dz} = \frac{1}{r}\frac{d}{dr}\left(r\mu(T)\frac{du}{dr}\right) \tag{2.1}$$

energy

$$\frac{d}{dr}\left(rk(T)\frac{dT}{dr}\right) = -r\mu(T)\left(\frac{du}{dr}\right)^2 \tag{2.2}$$

where $T = T(r)$ and $u = u(r)$.

In dimensionless form the momentum equation becomes

$$R^2\left(\frac{dP}{dz}\right) = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{\mu_o}{\bar{T}^2}\frac{T_o^2}{T_1^2}u_m\frac{d\bar{u}}{d\rho}\right) \tag{2.3}$$

where $r = \rho R$, $T = \bar{T}T_1$, and $u = \bar{u}u_m$. The constant u_m represents the maximum velocity achieved by the system. After one integration equation (2.3) becomes

$$\frac{d\bar{u}}{d\rho} = \frac{R^2T_1^2}{2\mu_o u_m T_o^2}\left(\frac{dP}{dz}\right)\rho\bar{T}^2 + \frac{C_1\bar{T}^2}{\rho}. \tag{2.4}$$

Since $d\bar{u}/d\rho = 0$ when $\rho = 0$, $C_1 = 0$.

Thus,

$$\frac{d\bar{u}}{d\rho} = \frac{R^2T_1^2}{2\mu_o u_m T_o^2}\left(\frac{dP}{dz}\right)\rho\bar{T}^2. \tag{2.5}$$

Where T_o is defined as $(T_1 + T_m)/2$ and T_m is the maximum temperature achieved by the system. Consider now the energy equation in dimensionless form with (2.5) substituted into the right-hand side of (2.2).

$$\frac{d}{d\rho}\left(\frac{\rho}{\bar{T}}\frac{d\bar{T}}{d\rho}\right) = \frac{-\Gamma}{(\lambda+1)^3}\rho^3\bar{T}^2 \tag{2.6}$$

where $\lambda = T_m/T_1$ and Γ is a flow parameter of the system given by:

$$\Gamma = \frac{2R^4}{\mu_o k_o T_1}\left(\frac{dP}{dz}\right)^2. \tag{2.7}$$

One integration of (2.6) leads to

$$\frac{d\bar{T}}{d\rho} = -\frac{\bar{T}}{\rho}\frac{\Gamma}{(\lambda+1)^3}\int_0^\rho \rho^3\bar{T}^2\,d\rho + C_2\frac{\bar{T}}{\rho}. \tag{2.8}$$

Since $d\bar{T}/d\rho = 0$ at $\rho = 0$, $C_2 = 0$.

Thus,

$$d\bar{T} = -\frac{\bar{T}}{\rho}\frac{\Gamma}{(1+\lambda)^3}\int_0^\rho \rho_3\bar{T}^2\,d\rho \tag{2.9}$$

or, upon integrating (2.9),

$$\bar{T} = \frac{-\Gamma}{(1+\lambda)^3}\int_0^\rho \frac{\bar{T}}{\rho}\left\{\int_0^\rho \rho^3\bar{T}^2\,d\rho\right\}d\rho + C_4 \tag{2.10}$$

with $\bar{T}(0) = \lambda$, $C_4 = \lambda$; thus,

$$\bar{T} = -\frac{\Gamma}{(1+\lambda)^3}\int_0^\rho \frac{\bar{T}}{\rho}\left\{\int_0^\rho \rho^3\bar{T}^2\,d\rho\right\}d\rho + \lambda. \tag{2.11}$$

Equation (2.11) is in a form amenable to solution by Picard's method of successive approximations. Consider (2.11) as

$$\bar{T} = \frac{-\Gamma}{(1+\lambda)^3}\int_0^\rho \frac{\bar{T}}{\epsilon}\left\{\int_0^\epsilon \xi^3\bar{T}^2\,d\xi\right\}d\epsilon + \lambda. \tag{2.12}$$

In the Picard's solution an initial guess, usually λ, is taken as $\bar{T}^{(0)}$. $\bar{T}^{(1)}$ is obtained by substituting $\bar{T}^{(0)}$ in the integrand of (2.12); that result is re-substituted into (2.12) to yield $\bar{T}^{(2)}$ and so on. The nth approximation is given by

$$\bar{T}^{(n)} = \frac{-\Gamma}{(1+\lambda)^3}\int_0^\rho \frac{\bar{T}^{(n-1)}}{\epsilon} \times\left\{\int_0^\epsilon \xi^3\bar{T}^{(n-1)}\,d\xi\right\}d\epsilon + \lambda. \tag{2.13}$$

In the case considered here the successive approximations are truncated at $\bar{T}^{(2)}$; thus,

$$\bar{T} \sim \bar{T}^{(2)} = \frac{\Gamma^4\lambda^9\rho^{16}}{786432\,(1+\lambda)^{12}} - \frac{\Gamma^3\lambda^7\rho^{12}}{9216\,(1+\lambda)^9} + \frac{\Gamma^2\lambda^5\rho^8}{256\,(1+\lambda)^6} - \frac{\Gamma\lambda^3\rho^4}{16\,(1+\lambda)^3} + \lambda. \quad (2.14)$$

The boundary condition $\bar{T}(\pm 1) = 1$ permits evaluation of the relationship between the flow parameter Γ and the temperature ratio λ as

$$\Gamma^4\lambda^9 - 85{\cdot}33\,\Gamma^3\lambda^7\,(1+\lambda)^3 + 3072\,\Gamma^2\lambda^5\,(1+\lambda)^6 - 49152\,\Gamma\lambda^3\,(1+\lambda)^9 + 786432\,(1+\lambda)^{12}\,(\lambda - 1) = 0. \quad (2.15)$$

dimensionless parameters as the Picard solution. The general functional form (2.4) may be expressed as

$$J = \int_0^{2\pi}\int_0^R \left\{\frac{k_oT_o}{2T^2}\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)^2 + \frac{\mu_oT_o^2}{2T^2}\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2 + u\frac{\mathrm{d}P}{\mathrm{d}z}\right\}\mathrm{d}r\,\mathrm{d}\theta \quad (2.18)$$

where the Cartesian coordinates of (2.4) easily transform to the coordinates r, z.

The velocity is obtained by solving (2.5)

$$\bar{u} = \frac{R^2T_1^2}{2\mu_ou_mT_o^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)\left[\frac{\Gamma^8\lambda^{18}\rho^{34}}{34.2^{36}.9.(1+\lambda)^{24}} - \frac{\Gamma^7\lambda^{16}\rho^{30}}{30.2^{27}.3.(1+\lambda)^{21}} + \frac{59\Gamma^6\lambda^{14}\rho^{26}}{26.2^{25}.3^4.(1+\lambda)^{18}} - \frac{19\Gamma^5\lambda^{12}\rho^{22}}{22.2^{24}.9.(1+\lambda)^{15}} + \frac{77\Gamma^4\lambda^{18}\rho^{18}}{18.2^{18}.9.(1+\lambda)^{12}} - \frac{13\Gamma^3\lambda^8}{7.2^{12}.3^2.(1+\lambda)^9} + \frac{3\Gamma^2\lambda^6}{10.2^8.(1+\lambda)^6} - \frac{\Gamma}{48(1+\lambda)^3} + \frac{\lambda^2}{2}\right] + 1. \quad (2.16)$$

The constant u_m is evaluated by using the condition $\bar{u}(\pm 1) = 0$. Introducing a dimensionless velocity u^* defined by

$$u^* = u\left(\sqrt{\frac{\mu_o}{k_oT_1}}\right)$$

where $u = u_m\bar{u}$

equation (2.16) may be written

$$u^* = \frac{-(\sqrt{2\Gamma})}{(1+\lambda)^2}\left\{\frac{\Gamma^8\lambda^{16}(\rho^{34}-1)}{34.2^{36}.9.(1+\lambda)^{24}} - \frac{\Gamma^7\lambda^{16}(\rho^{30}-1)}{30.2^{27}.9.(1+\lambda)^{15}} + \frac{59\Gamma^6\lambda^{14}(\rho^{26}-1)}{26.2^{25}.3^4.(1+\lambda)^{18}} - \frac{19\Gamma^5\lambda^{12}(\rho^{22}-1)}{22.2^{21}.9.(1+\lambda)^{15}} + \frac{77\Gamma^4\lambda^{10}(\rho^{18}-1)}{18.2^{18}.9.(1+\lambda)^{12}} - \frac{13\Gamma^3\lambda^8(\rho^{14}-1)}{7.2^{12}.9.(1+\lambda)^9} + \frac{3\Gamma^2\lambda^6(\rho^{10}-1)}{10.2^8.(1+\lambda)^6} - \frac{\Gamma(\rho^6-1)}{48(1+\lambda)^3} - \frac{\lambda^2(\rho^2-1)}{2}\right\}. \quad (2.17)$$

Thus, for a given value of Γ, the temperature ratio λ can be found from (2.15) and the temperature and velocity distributions may be found from (2.14) and (2.17).

Variational solution

The variational solution to the above problem makes use of the same system geometry and

Under the transformation $\rho = (r/R)$ where $0 \leqslant \rho \leqslant 1$ the functional (2.12) becomes

$$J = \int_0^{2\pi}\int_0^1 \left\{\frac{k_oT_o}{2T^2}\left(\frac{\mathrm{d}T}{\mathrm{d}\rho}\right)^2 + \frac{\mu_oT_o^2}{2T^2}\left(\frac{\mathrm{d}u}{\mathrm{d}\rho}\right)^2 + R^2u\frac{\mathrm{d}P}{\mathrm{d}z}\right\}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta \quad (2.19)$$

with boundary conditions:

$$T(\pm 1) = T_1 \qquad T(0) = T_m$$
$$u(\pm 1) = 0 \qquad u(0) = u_m$$

where u_m and T_m are the maximum values of velocity and temperature. Using the Rayleigh–Ritz procedure [4] the assumed functions of temperature and velocity are taken as,

$$T = T_m - (T_m - T_1)\rho^4 \tag{2.20}$$

and

$$u = u_m(1 - \rho^2). \tag{2.21}$$

The arbitrary constants are evaluated from the relationships $\partial J/\partial \alpha_i = 0$ where $\alpha_1 = T_m$, $\alpha_2 = u_m$.
Thus,

The integration becomes elementary if the substitution $K^2 - \rho^4 = \xi$ is made. Thus, after simple algebraic manipulation and integration (2.24) becomes

$$\left[\frac{K^4 - K^2}{2\xi^2} - \frac{K^2 - 1}{\xi} + C_o\left(\frac{K^2 - 1}{2\xi^2} - \frac{1}{\xi}\right)\right]_{K^2}^{K^2 - 1} = 0$$

which reduces to

$$C_o = K^2 = \frac{\lambda}{\lambda - 1}$$

or, using the definition of C_o

$$u_m^2 = 4\left(\frac{\lambda}{\lambda - 1}\right)\frac{\Delta^2 k_o}{\mu_o T_o}. \tag{2.25}$$

$$\frac{\partial J}{\partial T_m} = 2\pi \int_0^1 \left\{\frac{k_o T_o}{2T^2} 2\left(\frac{\partial T}{\partial \rho}\right)\frac{\partial}{\partial T_m}\left(\frac{\partial T}{\partial \rho}\right) + \left(\frac{\partial T}{\partial \rho}\right)^2 \frac{k_o T_o}{2}\left(-\frac{2}{T^3}\frac{\partial T}{\partial T_m}\right) + \frac{\mu_o T_o^2}{2}\left(\frac{\partial u}{\partial \rho}\right)^2\left(-\frac{2}{T^3}\frac{\partial T}{\partial T_m}\right)\right\}\rho\, \mathrm{d}\rho = 0$$

or,

$$\int_0^1 \left\{\frac{k_o T_o}{T^2}\frac{\partial T}{\partial \rho}\frac{\partial}{\partial T_m}\left(\frac{\partial T}{\partial \rho}\right) - \frac{k_o T_o}{T^3}\left(\frac{\partial T}{\partial \rho}\right)^2\left(\frac{\partial T}{\partial T_m}\right) - \frac{\mu_o T_o^2}{T^3}\left(\frac{\partial u}{\partial \rho}\right)^2\left(\frac{\partial T}{\partial T_m}\right)\right\}\rho\, \mathrm{d}\rho = 0. \tag{2.22}$$

Substitution of the appropriate derivatives into (2.22) yields

$$\int_0^1 \left\{\frac{k_o T_o}{[T_m - (T_m - T_1)\rho^4]^2}[-4(T_m - T_1)\rho^3](-4\rho^3) - \frac{k_o T_o}{[T_m - (T_m - T_1)\rho^4]^3} \times [-4(T_m - T_1)]^2(1 - \rho^4) - \frac{\mu_o T_o^2}{[T_m - (T_m - T_1)\rho^4]^3}(-2u_m\rho)(1 - \rho^4)\right\}\rho\, \mathrm{d}\rho \tag{2.23}$$

introduction of the quantities

$$K^2 = \frac{T_m}{T_m - T_1} = \frac{\lambda}{\lambda - 1}$$

and

$$\Delta = (T_m - T_1), \qquad C_o = \frac{\mu_o T_o u_m^2}{4\Delta^2 k_o}$$

reduces the above relation to the simpler form

$$\int_0^1 \left\{\frac{\rho^7}{(K^2 - \rho^4)} - \frac{\rho^7(1 - \rho^4)}{(K^2 - \rho^4)^3} - \frac{C_o\rho^3(1 - \rho^4)}{(K^2 - \rho^4)^3}\right\}\mathrm{d}\rho = 0. \tag{2.24}$$

In similar fashion

$$\frac{\partial J}{\partial u_m} = 2\pi \int_0^1 \left\{ \frac{\mu_o T_o^2}{T^2} \left(\frac{\partial u}{\partial \rho} \right) \frac{\partial}{\partial u_m} \left(\frac{\partial u}{\partial \rho} \right) + R^2 \left(\frac{\mathrm{d}P}{\mathrm{d}z} \right) \frac{\partial u}{\partial u_m} \right\} \rho \, \mathrm{d}\rho = 0. \qquad (2.26)$$

Evaluating the required derivatives and integrating leads to

$$u_m = -\frac{R^2}{4} \left(\frac{\mathrm{d}p}{\mathrm{d}z} \right) \frac{\Delta^2}{\mu_o T_o^2} \frac{\lambda}{(\lambda - 1)^2}$$

or

$$u_m^2 = \frac{R^4}{16} \left(\frac{\mathrm{d}p}{\mathrm{d}z} \right)^2 \frac{\Delta^4}{\mu_o^2 T_o^4} \frac{\lambda^2}{(\lambda - 1)^4}. \qquad (2.27)$$

Equating (2.25) and (2.27) leads to

$$\Gamma = \frac{16(\lambda - 1)(\lambda + 1)^3}{\lambda} \qquad (2.28)$$

where Γ is defined by (2.7). The dimensionless temperature and velocity distributions become

$$\bar{T} = \lambda - (\lambda - 1)\rho^4 \qquad (2.29)$$

$$u^* = \left(\sqrt{\frac{\Gamma}{2}} \right) \frac{\lambda}{(\lambda + 1)^2} (1 - \rho^2) \qquad (2.30)$$

where $\bar{T}$ and u^* are defined as in the Picard's solution. Thus, for a given value of the flow parameter Γ the temperature ratio λ is available from (2.28) leaving the temperature distribution (2.29) and the velocity distribution (2.30) explicit functions of the independent variable ρ.

3. COUETTE FLOW

Closed form solution

The second flow system under study is Couette flow between two infinite parallel plates separated by a flow height h. The lower plate is stationary and the upper plate moves with a constant velocity U. The system is illustrated in Fig. 2. Both plates are kept at a constant temperature T_1 and the flow is unidimensional. The first solution is the closed-form solution.

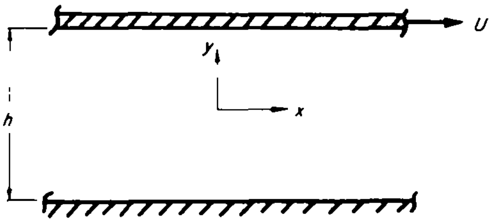

FIG. 2. Geometry of system for Couette flow.

The introduction of the dimensionless quantities

$$\phi = \frac{2y}{r} \qquad \bar{T} = \frac{T}{T_1} \qquad \bar{u} = \frac{u}{U}$$

and evaluation of the phenomenological coefficients μ_o, and k_o, at a reference temperature T_o defined by

$$T_o = \frac{T_1 + T_m}{2}$$

where T_m is the maximum temperature achieved in the system, reduces the energy equation to

$$-\frac{\mathrm{d}}{\mathrm{d}\phi} \left(\frac{1}{\bar{T}} \frac{\mathrm{d}\bar{T}}{\mathrm{d}\phi} \right) = \chi \left(1 + \frac{T_m}{T_1} \right) \frac{1}{\bar{T}^2} \left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}\phi} \right)^2 \qquad (3.1)$$

where

$$\chi = \frac{\mu_o U^2}{2K_o T_1}.$$

Likewise the momentum equation is reduced to

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}\phi} = \alpha \bar{T}^2 \qquad (3.2)$$

after the required substitutions are made and one integration is performed. The constant α is a constant of integration. If (3.2) is substituted into (3.1) and the indicated differentiation is performed the following equation is obtained

$$\bar{T} \frac{\mathrm{d}^2 \bar{T}}{\mathrm{d}\phi^2} - \left(\frac{\mathrm{d}\bar{T}}{\mathrm{d}\phi} \right) + \gamma \bar{T}^4 = 0 \qquad (3.3)$$

where

$$\gamma = \alpha^2 \chi \left(1 + \frac{T_m}{T_1} \right).$$

The use of the substitution

$$z(\bar{T}) = \left(\frac{\mathrm{d}\bar{T}}{\mathrm{d}\phi}\right)^2$$

reduces (3.3) to

$$\bar{T}\frac{\mathrm{d}z}{\mathrm{d}\bar{T}} - 2z + 2\gamma\bar{T}^4 = 0. \quad (3.4)$$

The transformation $\xi = \bar{T}^2$ and $z = \bar{T}^2 f(\xi)$ transforms (3.4) to

$$\frac{\mathrm{d}}{\mathrm{d}\xi} f(\xi) = -\gamma \quad (3.5)$$

or

$$f(\xi) = -\gamma\xi + C_1. \quad (3.6)$$

In terms of the original variables (3.6) becomes

$$\left(\frac{\mathrm{d}\bar{T}}{\mathrm{d}\phi}\right)^2 = -\gamma\bar{T}^4 + C_1\bar{T}^2 \quad (3.7)$$

which can be written in the form

$$\frac{\mathrm{d}\bar{T}}{\mathrm{d}\phi} = i(\sqrt{\gamma})\,\bar{T}\sqrt{(\bar{T} - C_1/\gamma)} \quad (3.8)$$

or, after integration,

$$\frac{-i}{\sqrt{(C_1/\gamma)}}\cos^{-1}\frac{\sqrt{(C_1/\gamma)}}{\bar{T}} = -(\sqrt{\gamma})\,\phi + C_2 \quad (3.9)$$

using the relation $-i$ arccos ω = arccosh ω equation (3.9 becomes

$$\bar{T} = \left(\sqrt{\frac{C_1}{\gamma}}\right)\operatorname{sech}\left\{[(\sqrt{\gamma})\phi + C_2]\left(\sqrt{\frac{C_1}{\gamma}}\right)\right\} \quad (3.10)$$

Two boundary conditions are that

$$\frac{\mathrm{d}\bar{T}}{\mathrm{d}\phi} = 0 \quad \text{when} \quad \bar{T} = \frac{T_m}{T_1} = \frac{1}{\eta} \quad (3.11)$$

and

$$\bar{T}(0) = \frac{1}{\eta} \quad (3.12)$$

where η is defined by $\eta = (T_1/T_m)$.

Both (3.11) and (3.12) are consequences of the symmetry of the system. Using (3.12) C_2 is zero, and using (3.11) $C_1 = (\gamma/\eta^2)$. Thus,

$$\bar{T} = \frac{1}{\eta}\operatorname{sech}\left[\frac{1}{\eta}(\sqrt{\gamma})\phi\right]. \quad (3.13)$$

Substitution of the value of the constant C_1 yields the following expression

$$\bar{T} = \frac{1}{\eta}\operatorname{sech}\left\{\frac{\alpha\sqrt{[\chi(\eta+1)]}}{\eta^{\frac{3}{2}}}\phi\right\}. \quad (3.14)$$

When the expression (3.14) is substituted into (3.2) and one integration is performed, the resulting expression for $\bar{u}$ is obtained.

$$\bar{u} = \frac{1}{\sqrt{[\eta\chi(\eta+1)]}}\tanh\left\{\frac{\alpha\sqrt{[\chi(\eta+1)]}}{\eta^{\frac{3}{2}}}\phi\right\} + C_3. \quad (3.15)$$

Since $\bar{u}(-1) = 0$, the constant can be evaluated to give

$$\bar{u} = \frac{1}{\sqrt{[\eta\chi(\eta+1)]}}\tanh\left\{\frac{\alpha\sqrt{[\chi(\eta+1)]}}{\eta^{\frac{3}{2}}}\phi\right\} + \frac{1}{\sqrt{[\eta\chi(\eta+1)]}}\tanh\left\{\frac{\alpha\sqrt{[\chi(\eta+1)]}}{\eta^{\frac{3}{2}}}\right\}. \quad (3.16)$$

The value of the constant of integration α is now determined from the boundary condition $\bar{u}(1) = 1$). Thus,

$$\alpha = \frac{\eta^{\frac{3}{2}}}{\sqrt{[\chi(\eta+1)]}}\tanh^{-1}\left\{\frac{\sqrt{[\eta\chi(\eta+1)]}}{2}\right\}. \quad (3.17)$$

The relationship $\chi(\eta)$ is determined from the boundary condition $\bar{T}(\pm 1) = 1$. Thus,

$$\chi = 4\left(\frac{1}{\eta} - 1\right). \quad (3.18)$$

Final relations for $\bar{T}$ and $\bar{u}$ are obtained when the value of χ from (3.18) is substituted into (3.14) and (3.16), or

$$\bar{u} = \frac{1}{2\sqrt{(1-\eta^2)}}\tanh[\phi\tanh^{-1}\sqrt{(1-\eta^2)}] + \tfrac{1}{2} \quad (3.19)$$

$$\bar{T} = \frac{1}{\eta}\operatorname{sech}[\phi\tanh^{-1}\sqrt{(1-\eta^2)}]. \quad (3.20)$$

Variational solution

As in the closed-form solution to Couette flow, $T = T(y)$ and $u = u(y)$. The body force is neglected and the normal components of the stress tensor are everywhere zero on the boundary. The functional to be minimized takes the form,

$$J = \int_{-1}^{1} \left\{ \frac{k_o T_o}{2T^2}\left(\frac{\mathrm{d}T}{\mathrm{d}\phi}\right)^2 + \frac{\mu_o T_o^2}{2T^2}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 \right\} \mathrm{d}\phi. \quad (3.21)$$

Using the Rayleigh–Ritz procedure the assumed temperature profile is given by

$$T = T_m - (T_m - T_1)\phi^2 \quad (3.22)$$

and the assumed velocity profile by

$$u = \frac{U}{2}(1 + \phi) + B\phi(1 - \phi^2) \quad (3.23)$$

where T_m and B are arbitrary constants. The boundary conditions are

$$T(\pm 1) = T_1 \qquad T(0) = T_m$$
$$u(-1) = 0 \qquad u(1) = U.$$

The arbitrary constant B is evaluated from the relationship

$$\frac{\partial J}{\partial B} = \int_{-1}^{1} \frac{\mu_o T_o^2}{T^2} \frac{\partial u}{\partial \phi} \frac{\partial}{\partial B}\left(\frac{\partial u}{\partial \phi}\right) \mathrm{d}\phi = 0. \quad (3.24)$$

After the required derivatives are evaluated and substituted into (3·24), and the indicated integration is performed, the following relationship is found.

$$(\tfrac{1}{2} + \beta)\left[\frac{1}{K^2(K^2 - 1)} + \frac{1}{2K^3}\ln\frac{K+1}{K-1}\right] - (\tfrac{3}{2} + 6\beta)\left[\frac{1}{K^2 - 1} - \frac{1}{2K}\ln\frac{K+1}{K-1}\right] + 9\beta\left[2 + \frac{K^2}{K^2 - 1} - \frac{3K}{4}\ln\frac{K+1}{K-1}\right] = 0 \quad (3.25)$$

where

$$\Delta = T_m(1 - \eta), \qquad K^2 = \frac{1}{1 - \eta},$$
$$\beta = \frac{B}{U} \quad \text{and} \quad \eta = \frac{T_1}{T_m}.$$

Equation (3.25) can be solved for β in terms of η. Thus,

$$\beta = \frac{\left[\dfrac{(2 + \eta)(1 - \eta)}{2\eta} + (\eta - 4)(1 - \eta)^{\frac{1}{2}} \ln\dfrac{1 + \sqrt{(1 - \eta)}}{1 - \sqrt{(1 - \eta)}}\right]}{\left[\dfrac{(\eta + 2)^2}{\eta} + 18 + \dfrac{(2\eta^2 - 16\eta - 13)}{4(1 - \eta)^{\frac{1}{2}}} \ln\dfrac{1 + \sqrt{(1 - \eta)}}{1 - \sqrt{(1 - \eta)}}\right]}. \quad (3.26)$$

The determination of the constant T_m is found by a similar procedure.

$$\frac{\partial J}{\partial T_m} = \int_{-1}^{1} \left\{ \frac{k_o T_o}{2}\left(\frac{\mathrm{d}T}{\mathrm{d}\phi}\right)^2 \frac{\partial}{\partial T_m}\left(\frac{1}{T^2}\right) + \frac{k_o T_o}{T^2}\frac{\mathrm{d}T}{\mathrm{d}\phi}\frac{\partial}{\partial T_m}\left(\frac{\partial T}{\partial \phi}\right) + \left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 \frac{\mu_o T_o^2}{2}\frac{\partial}{\partial T_m}\left(\frac{1}{T^2}\right)\right\} \mathrm{d}\phi = 0. \quad (3.27)$$

Equation (3.27) may be rewritten as

$$\int_{-1}^{1} \left\{ \frac{\phi^4}{(K^2 - \phi^2)^3} - \frac{\phi^2}{(K^2 - \phi^2)^3} + \frac{\phi^2}{(K^2 - \phi^2)^2} - \frac{\chi\eta(1 + \eta)}{4(1 - \eta)^2}\left[\frac{(\frac{1}{2} + \beta - 3\beta\phi^2)(1 - \phi^2)}{(K^2 - \phi^2)^3}\right]\right\} \mathrm{d}\phi = 0 \quad (3.28)$$

where previously defined dimensionless quantities are employed. After integration (3.28) takes the form

$$\frac{(2-\eta)(1-\eta)}{4\eta} - \frac{\eta(1-\eta)^{\frac{1}{2}}}{8}\ln\frac{1+\sqrt{(1-\eta)}}{1-\sqrt{(1-\eta)}} - \frac{\chi\eta(1+\eta)}{4(1-\eta)^2}\left\{\frac{(1-\eta)^2(2-\eta)}{16\eta}\right.$$
$$+ \frac{(4-3\eta)(1-\eta)^{\frac{1}{2}}}{32}\ln\frac{1+\sqrt{(1-\eta)}}{1-\sqrt{(1-\eta)}} + \beta\left[\frac{(1-\eta)(\eta^2-2\eta-10)}{4\eta}\right.$$
$$\left. + \frac{16-10\eta+3\eta^2}{8}(1-\eta)^{\frac{1}{2}}\ln\frac{1+\sqrt{(1-\eta)}}{1-\sqrt{(1-\eta)}}\right] + \beta^2\left[\frac{(1-\eta)(36-91\eta-5\eta^2+\eta^3)}{4\eta^2} - 18\right.$$
$$\left.\left. + \frac{(190-68\eta+16\eta^2-3\eta^3)}{8(1-\eta)^{\frac{1}{2}}}\ln\frac{1+\sqrt{(1-\eta)}}{1-\sqrt{(1-\eta)}}\right]\right\} = 0. \qquad (3.29)$$

The dimensionless temperature and velocity functions become

$$\bar{T} = \frac{1}{\eta}[1-(1-\eta)\phi^2] \qquad (3.30)$$

and

$$\bar{u} = \tfrac{1}{2}(1+\phi) + \beta\phi(1-\phi^2). \qquad (3.31)$$

Thus, relations (3.26) and (3.29) may be solved numerically for β and η for various values of χ and the relations (3.30) and (3.31) may be used for obtaining velocity and temperature distributions.

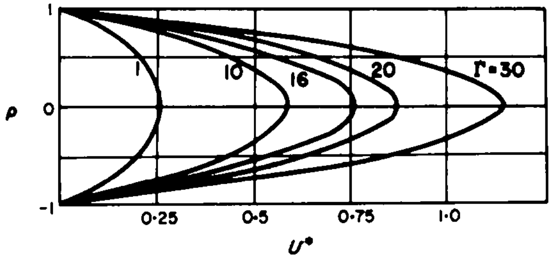

FIG. 3. Velocity distributions in Poiseuille flow.

4. RESULTS

Poiseuille flow

The relationship between the flow parameter Γ and the temperature ratio λ for the Picard solution is given in (2.15) and for the variational solution in (2.28). Calculations are made by first selecting a value for the flow parameter Γ, and then solving the resulting relations (2.15) and (2.28) for the roots $\lambda > 1$. Equations (2.15) and (2.16) take the form $F(\lambda) = 0$ after substitution of a value for Γ. The root of $F(\lambda) = 0$ was found by the method of bisection using a digital computer. This method consists of evaluating the function $F(\lambda)$ at an arbitrary starting point λ_0 and then at successive intervals of $\Delta\lambda$ noting the interval where $F(\lambda)$ changes sign. The interval is then halved until the root is found to a specified accuracy ϵ. A value of 0·001 for ϵ proved quite sufficient. After the roots of $F(\lambda) = 0$ are found for various values of Γ, the temperature and velocity distributions may be calculated by substitution into relations (2.14) and (2.17) for the Picard solution, and relations (2.29) and (2.30) for the variational solution.

The actual profiles for both temperature and velocity for the variational solution are sufficiently close to those of the Picard solution as to prevent accurate plotting. For this reason, only the Picard solution for velocity and temperature distributions are plotted in Figs. 3 and 4. Close comparisons of the variational and Picard solutions may be made by examining

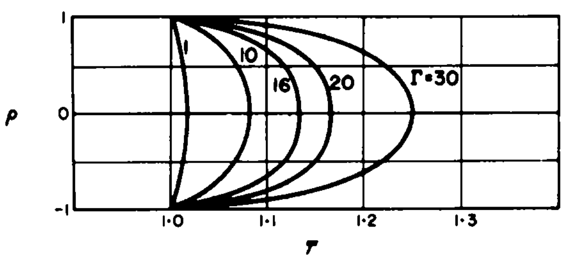

FIG. 4. Temperature distributions in Poiseuille flow.

the detailed data presented in Table 1. A graph of per cent temperature variation vs. the non-dimensional radius is presented in Fig. 5 and a similar graph is presented in Fig. 6 for per cent velocity variation.

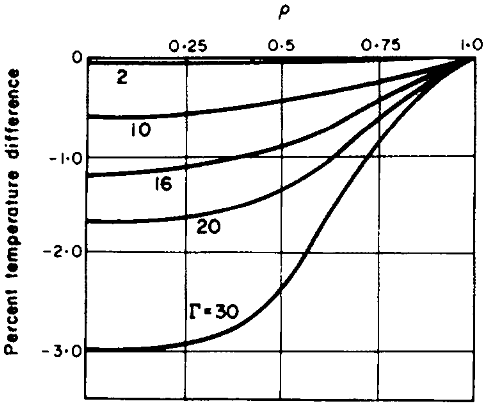

FIG. 5. Per cent temperature difference for a variational solution to Poiseuille flow based on a Picard approximation.

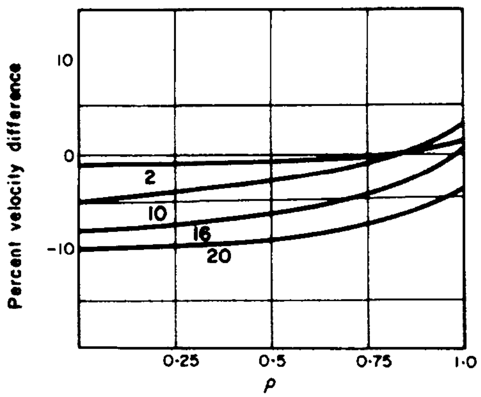

FIG. 6. Percent velocity difference for a variational solution to Poiseuille flow based on a Picard approximation.

It is apparent from the previous graphs and tables that the solution for the temperature distribution by the variational method agrees very well with the solution using Picard's method for all but the largest values of the flow parameter Γ. The error for the velocity distribution increases with higher values of Γ, however the error approaches 10 per cent for a value of $\Gamma = 20$ which corresponds to a very high

Table 1. Dimensionless temperature and velocity profiles as a function of the flow parameter and the radius ρ as obtained by Picard's (p) and a variational formulation (v)

Γ	ρ	$\bar{T}(v)$	$\bar{T}(p)$	$u^*(v)$	$u^*(\partial)$
2.	0·0	1·0155	1·0157	0·2499	0·2514
	0·1	1·0155	1·0157	0·2475	0·2489
	0·2	1·0155	1·0157	0·2399	0·2413
	0·3	1·0154	1·0156	0·2275	0·2286
	0·4	1·0151	1·0153	0·2000	0·2108
	0·5	1·0145	1·0147	0·1875	0·1880
	0·6	1·0135	1·0136	0·1600	0·1602
	0·7	1·0117	1·0119	0·1275	0·1273
	0·8	1·0092	1·0092	0·0900	0·0896
	0·9	1·0053	1·0053	0·0475	0·0471
	1·0	1·0000	1·0000	0·0000	0·0000
16.	0·0	1·1177	1·1307	0·7049	0·7612
	0·1	1·1177	1·1307	0·6978	0·7533
	0·2	1·1175	1·1305	0·6767	0·7294
	0·3	1·1167	1·1294	0·6415	0·6896
	0·4	1·1114	1·1269	0·5921	0·6340
	0·5	1·1103	1·1121	0·5287	0·5628
	0·6	1·1102	1·1117	0·4512	0·4764
	0·7	1·0894	1·0959	0·3595	0·3758
	0·8	1·0695	1·0727	0·2538	0·2618
	0·9	1·0405	1·0408	0·1339	0·1361
	1·0	1·0000	1·0000	0·0000	0·0000

Table 2. Dimensionless temperature and velocity profiles as a function of the flow parameter χ and the dimensionless space coordinates φ as obtained from an exact solution (e) and a variational formulation (v)

χ	ϕ	$T(v)$	$T(e)$	$\bar{u}(v)$	$\bar{u}(e)$
0·5	−1·0	1·0000	1·0000	0·0000	0·0000
	−0·8	1·0750	1·0422	0·0951	0·0891
	−0·6	1·1133	1·0772	0·1935	0·1951
	−0·4	1·1751	1·1033	0·2943	0·2867
	−0·2	1·2001	1·1195	0·3967	0·3923
	0·0	1·2084	1·1250	0·5000	0·5000
	0·2	1·2001	1·1195	0·6033	0·6077
	0·4	1·1751	1·1033	0·7070	0·7133
	0·6	1·1133	1·0772	0·8065	0·8149
	0·8	1·0750	1·0422	0·9049	0·9109
	1·0	1·0000	1·0000	1·0000	1·0000
4·0	−1·0	1·0000	1·0000	0·0000	0·0000
	−0·8	1·3424	1·2438	0·0888	0·4788
	−0·6	1·6087	1·5053	0·1851	0·1199
	−0·4	1·7990	1·7514	0·2869	0·2212
	−0·2	1·9132	1·9326	0·3925	0·3513
	0·0	1·9512	2·0000	0·5000	0·5000
	0·2	1·9132	1·9326	0·6074	0·6486
	0·4	1·7990	1·7514	0·7131	0·7788
	0·6	1·6087	1·5053	0·8149	0·8801
	0·8	1·3424	1·2438	0·9112	0·9521
	1·0	1·0000	1·0000	1·0000	1·0000

flow rate. Since the percent error is based on the Picard method, the comparison is made between two approximate solutions. However, a comparison to the work of Hays [3] indicates that the Picard's approximation to the exact solution is reasonably accurate.

Couette flow

The relationship between the flow parameter χ for Couette flow and the temperature ratio η is given for the closed form solution by the simple relation (3.18). For a given value for χ, the temperature and velocity and temperature functions (3.19) and (3.20) become functions of the dimensionless coordinate ϕ alone. The $\chi - \eta$ relationship for the variational solution is a complicated form requiring both relations (3.26) and (3.29). Recourse to the digital computer must be made to find the temperature ratios η for given values of the flow parameter χ. The bisection method described in the previous section is used for this task. Once the root $\eta < 1{\cdot}0$ is found, the temperature and velocity distributions are easily generated from equations (3.30) and (3.31).

Again close agreement between the variational and closed-form solutions prevent accurate comparison by plotting of temperature and velocity profiles. The temperature and velocity distributions as found from the exact solutions are shown in Figs. 7 and 8. Detailed data are presented in Table 2. A graph of percent temperature difference vs. the non-dimensional space coordinate ϕ is presented in Fig. 9 and the analogous graph for velocity data is presented in Fig. 10.

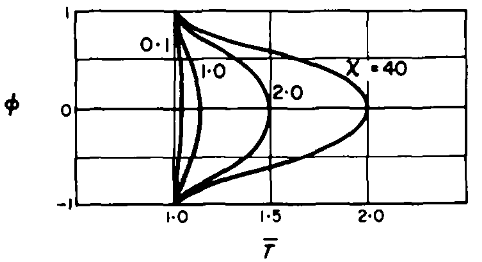

FIG. 7. Temperature difference in Couette flow.

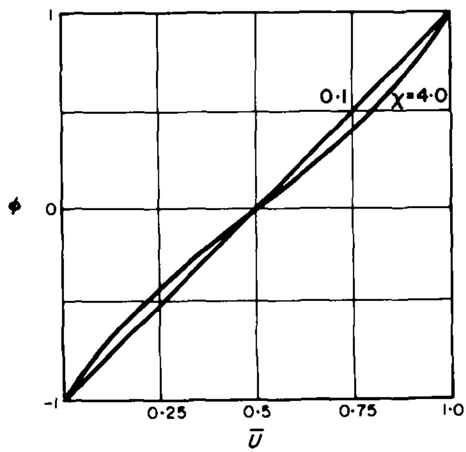

FIG. 8. Velocity difference in Couette flow.

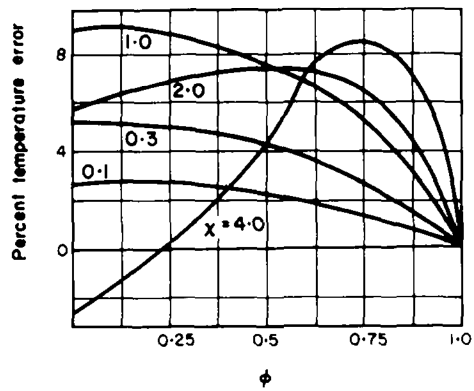

FIG. 9. Percent temperature error for variational solution based on exact solution to Couette flow.

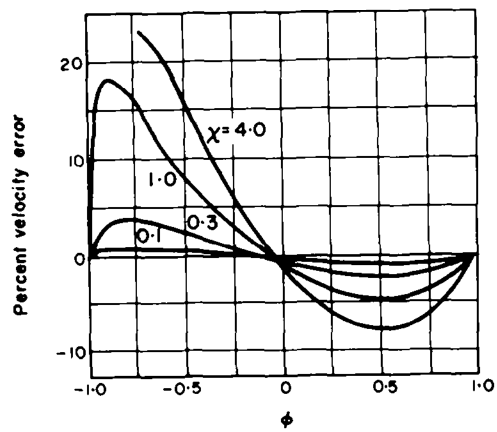

FIG. 10. Percent velocity error for variational solution based on exact solution to Couette flow.

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Résumé—Les articles récents de Glansdorff, Prigogine et Hays ont montré qu'un principe variationnel peut être appliqué aux problèmes de l'écoulement d'un fluide. Comme exemples de l'emploi d'une formulation variationnelle, les problèmes de l'écoulement incompressible visqueux à faible vitesse entre des plaques parallèles et dans un tube circulaire sont résolus dans le cas où les coefficients phénoménologiques de conductivité thermique et de viscosité sont fonctions de la température. La méthode de Rayleigh–Ritz est employée sous forme variationnelle pour obtenir des solutions qui sont comparées aux solutions obtenues par des techniques analytiques directes. On obtient un accord étroit entre les deux méthodes d'analyse pour les écoulements de Couette et Poiseuille, établissant ainsi un degré de confiance dans la solution variationnelle des problèmes pour lesquels des solutions directes ne peuvent pas être obtenues.

Zusammenfassung—Die neueren Arbeiten von Glansdorff, Prigogine und Hays haben gezeigt, dass ein Variationsprinzip auf Probleme der Flüssigkeitsströmung angewendet werden kann. Als Beispiele für Variationsformulierungen sind Probleme der langsamen, zähen, inkompressiblen Strömung zwischen parallelen Platten und in einem Rohr mit Kreisquerschnitt für den Fall gelöst, dass die phänomenologischen Koeffizienten der Wärmeleitung und der Zähigkeit, Funktionen der Temperatur sind. Die Methode von Rayleigh–Ritz wird mit der Variationsform verwendet, um Lösungen zu erhalten, die verglichen werden mit Lösungen, die sich durch direkte Analyse ergeben. Gute Übereinstimmung zwischen den beiden Methoden zeigt sich sowohl für Couette- als auch für Poiseuilleströmung, womit sich ein Zuverlässigkeitsmass für die Variationslösung ergibt, für die Probleme, die keine direkte Lösung erlauben.

Аннотация—В недавно опубликованных статьях Глансдорфа, Пригожина и Хейса показано, что вариационный принцип можно применить к задачам течения жидкости. Примерами применения вариационного метода являются задачи медленного течения вязкой несжимаемой жидкости между параллельными пластинами и в круглой трубе, которые решены для случая, когда феноменологические коэффициенты теплопроводности и вязкости являются функциями температуры. Метод Релея–Ритца в вариационном виде использовался для получения решений, которые сравниваются с решениями, полученными непосредственными аналитическими методами. Получено хорошее соответствие между этими двумя методами для течений Куэтта и Пуазейля; таким образом, установлена возможность применения вариационных методов к тем задачам, для которых точные решения не могут быть получены.